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Nonlinear Parametric Oscillations in Certain Stochastic Systems: A Random van der Pol Oscillator

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A *stochastic* model for some class of *nonlinear* oscillators, which includes a van der Pol-type oscillator with random parameters, is analyzed in the *diffusion limit.* That is, small random fluctuations and long time are considered, while the nonlinearity is also assumed to be small. We show that there exist *stationary distributions,* independent of the phase of the oscillator, a result proved earlier by R. L. Stratonovich assuming the random perturbations of the frequency to be delta correlated. The time behavior of the moments of the displacement of the oscillator from its rest position is also investigated and the results are compared with the corresponding ones for the linear random oscillator. A numerical study is also performed for the first two moments and plots are given.

KEY WORDS: Nonlinear random oscillators; van der Pol oscillator; initial-value problems for stochastic ordinary differential equations; numerical treatment of boundary-value problems for linear singular parabolic differential equations.

1. INTRODUCTION

An oscillator is a physical system where the oscillatory behavior, i.e., essentially the periodic behavior of some quantity, is the main feature. This includes such mechanical and electrical systems as pendulums, clocks, vacuum-tube devices, or just a particle performing a periodic motion around its rest position, subject to some restoring forces. Indeed, the atomic oscillator is one of the simplest examples, which is quite familiar to

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all physicists. To engineers, oscillator is a synonym of a circuit designed to generate sine waves (or other wave forms) for a variety of purposes such as, for instance, producing the carrier frequency in radio communications.

The oscillatory behavior referred to above may represent a wanted and valuable effect or also a parasitic or harmful one.

In the simplest case of systems with only one degree of freedom, the relevant mathematical description requires studying *initial-value* (IV) *problems* for *second-order* ordinary differential equations (ODE's).

In order to have "self-excited" (i.e., some kind of self-sustained) oscillations, say in some given electrical circuit, one is led to study certain classes of *nonlinear* second-order ODE's: for instance, the nonlinearity represents the saturating limiting effect, due to the device itself which, on the other hand, produces and amplifies some voltage or current (see Ref. 9, for example).

In general, some or all of the *parameters* of the system, i.e., of the coefficients of the ODE, *are time dependent*. This fact is referred to by saying that we are considering *parametric systems* (cf. Ref. 10, p. 277).

On the other hand, even *small* random fluctuations in the parameters, accumulated over *long* intervals of time, are important, in that they may change radically every situation for which a deterministic analysis predicts *stability or instability*. For example, referring again to the case of a vacuum-tube circuit, one may wish to take into account the effects of internal shot noise or that of externally applied random excitations.

In this paper we study *free self-excited* oscillations in *certain* nonlinear parametric systems with random parameters. They include a stochastic version of the classical van der Pol oscillator. It is when the characteristic of the vacuum tube is cubic that we are led to study the *van der Pol equation*, with various forms for the random coefficients (see, e.g., Ref. 10).

For generalities about *linear random* oscillators see, e.g., Refs. 1 and 6.

For the sake of generality, we shall consider the IV problem

$$\begin{gathered} y^{\varepsilon''} + 2\varepsilon^2[\lambda(t) + \nu(t)\, w(y^\varepsilon)^2]\, y^{\varepsilon'} + \omega_0^2[1 + \varepsilon\mu(t)]\, y^\varepsilon = 0 \\ y^\varepsilon(0) = y_1, \qquad y^{\varepsilon'}(0) = y_2 \end{gathered} \tag{1.1}$$

where y^ε is real valued, $\varepsilon > 0$ is a small parameter, w is a real parameter which sizes the nonlinearity, ω_0 is the radian frequency of the (unperturbed) oscillator; $\lambda(t)$, $\mu(t)$, $\nu(t)$ are suitable stochastic processes on some probability space.

This equation includes the classical deterministic van der Pol oscillator $[\lambda(t) \equiv -1,\ \nu(t)w \equiv 1,\ \mu(t) \equiv 0]$, while by replacing $(y^\varepsilon)^2$ by $(y^{\varepsilon'})^2$ we get the classical *Rayleigh* oscillator.

It is convenient to introduce the van der Pol variables $\rho^\varepsilon(t)$, $\phi^\varepsilon(t)$, defined by

$$\begin{aligned} y^\varepsilon &= \rho^\varepsilon(t)\cos[\omega_0 t + \phi^\varepsilon(t)] \\ y^{\varepsilon'} &= -\omega_0 \rho^\varepsilon(t)\sin[\omega_0 t + \phi^\varepsilon(t)] \end{aligned} \tag{1.2}$$

so that equation (1.1) can be rewritten in the form

$$\begin{cases} \rho^{\varepsilon'} = \varepsilon \dfrac{\omega_0}{2}\rho^\varepsilon \mu(t)\sin(2\omega_0 t + 2\phi^\varepsilon) \\ \qquad - \varepsilon^2 \rho^\varepsilon \left\{ \nu(t)\dfrac{w}{2}\rho^{\varepsilon^2}[1+\cos(2\omega_0 t + 2\phi^\varepsilon)] + \lambda(t)\right\} \\ \qquad \times [1-\cos(2\omega_0 t + 2\phi^\varepsilon)] \\ \phi^{\varepsilon'} = \varepsilon \dfrac{\omega_0}{2}\mu(t)[1+\cos(2\omega_0 t + 2\phi^\varepsilon)] \\ \qquad - \varepsilon^2 \left\{ \nu(t)\dfrac{w}{2}\rho^{\varepsilon^2}[1+\cos(2\omega_0 t + 2\phi^\varepsilon)] + \lambda(t)\right\} \sin(2\omega_0 t + 2\phi^\varepsilon) \end{cases} \tag{1.3}$$

with $\rho^\varepsilon(0) = (y_1^2 + \omega_0^{-2} y_2^2)^{1/2}$, $\phi^\varepsilon(0) = -\arctan(y_2/\omega_0 y_1)$.

We write, for short

$$\begin{cases} \rho^{\varepsilon'} = \varepsilon F_1(t; \rho^\varepsilon, \phi^\varepsilon) + \varepsilon^2 G_1(t; \rho^\varepsilon, \phi^\varepsilon) \\ \phi^{\varepsilon'} = \varepsilon F_2(t; \rho^\varepsilon, \phi^\varepsilon) + \varepsilon^2 G_2(t; \rho^\varepsilon, \phi^\varepsilon) \end{cases} \tag{1.4}$$

In Section 2 we state the main hypotheses on the processes λ, μ, ν and compute the infinitesimal generator for the limiting process to which the process $(\rho^\varepsilon, \phi^\varepsilon)$ converges, in some sense. Therefore, the Fokker–Planck and the Kolmogorov backward equations for such a limiting process are readily obtained. With such equations at hand, we prove the existence of nontrivial *stationary distributions*, independent of phase (Section 3): This result was proved earlier by Stratonovich, assuming the random perturbation of the frequency to be a *delta-correlated* process, as an approximation of a stochastic process with *small correlation time* (Ref. 10, pp. 302–305).

The time evolution of the moments of the displacement of the oscillator from its rest position is also studied (Section 4): We first discuss existence, uniqueness, and obtain some estimates from the backward equation, by using the Feynman–Kac formula. Finally, in Section 5, we perform a numerical study of the first two moments.

2. ASYMPTOTIC STATISTICAL ANALYSIS

We assume that $\lambda(t)$, $\mu(t)$, $\nu(t)$ in (1.3) are real-valued stochastic processes on some probability space $(\Omega, \mathcal{A}, P)$ almost surely bounded, wide-sense stationary and such that $E\{\lambda(t)\} = \gamma$, $E\{\mu(t)\} = 0$, $E\{\nu(t)\} = \nu_0 > 0$, $E\{\mu(t)\,\mu(s)\} = R(|t-s|)$. Here $E\{\cdot\}$ means taking expected values, that is integration over Ω, with respect to the measure P, and $R(\cdot)$ is the *correlation function* of $\mu(t)$. The process $\mu(t)$ is supposed to satisfy a *strong mixing* condition, with mixing rate decreasing to zero sufficiently fast (cf. Refs. 5 and 7).

Under these conditions, it is possible to carry out an asymptotic analysis of system (1.3), which allows us to compute various statistics of the solution $(\rho^\varepsilon(t), \phi^\varepsilon(t))$ and therefore of $y^\varepsilon(t)$. In fact, a limit theorem due to Khas'minskii[5] ensures that, for every $\tau_0 > 0$, the process $(\rho^\varepsilon(\varepsilon^2 t), \phi^\varepsilon(\varepsilon^2 t))$ converges weakly, as $\varepsilon \to 0$, $t \to +\infty$ with $\varepsilon^2 t = \text{const}$, (diffusion limit), on the interval $0 \leqslant \varepsilon^2 t < \tau_0$, to a *Markov process*, say $(\rho(\varepsilon^2 t), \phi(\varepsilon^2 t))$. Moreover, this process turns out to be continuous with probability 1, and with *infinitesimal generator*

$$L = \sum_{i,j=1}^{2} a_{ij}(\mathbf{z}) \frac{\partial^2}{\partial z_i\, \partial z_j} + \sum_{i=1}^{2} [b_i(\mathbf{z}) + c_i(\mathbf{z})] \frac{\partial}{\partial z_i} \tag{2.1}$$

where $\mathbf{z} := [\rho \phi]^T$, and the *diffusion matrix* $\{a_{ij}(\mathbf{z})\}$ and the *drift vector* $[b_i(\mathbf{z}) + c_i(\mathbf{z})]^T$ are given by

$$\begin{aligned}
a_{ij}(\mathbf{z}) &= \lim_{t \to +\infty} t^{-1} \int_{t_0}^{t_0+t} \int_{t_0}^{s} E\{F_i(s, \mathbf{z})\, F_j(\sigma, \mathbf{z})\}\, ds\, d\sigma \qquad (i, j = 1, 2)\\
b_i(\mathbf{z}) &= \lim_{t \to +\infty} t^{-1} \int_{t_0}^{t_0+t} \int_{t_0}^{s} \sum_{j=1}^{2} E\left\{F_j(s, \mathbf{z}) \frac{\partial F_i(\sigma, \mathbf{z})}{\partial z_j}\right\} ds\, d\sigma \qquad (2.2)\\
c_i(\mathbf{z}) &= \lim_{t \to +\infty} t^{-1} \int_{t_0}^{t_0+t} E\{G_i(s, \mathbf{z})\}\, ds \qquad (i = 1, 2)
\end{aligned}$$

with $F_i(s, \mathbf{z}) \equiv F_i(s; \rho, \phi)$, $G_i(s, \mathbf{z}) \equiv G_i(s; \rho, \phi)$. The integrals in (2.2) are supposed to exist uniformly in t_0 and $\mathbf{z}$.

Remark 2.1. We observe that F_i, G_i and their first two derivatives with respect to z_i do *not* satisfy the boundedness condition which was assumed in the original formulation of Khas'minskii's theorem. However, this theorem still holds true as long as the problem (2.9) below has a *unique* solution: a_{ij}, b_i, c_i can be *unbounded* provided that the limiting process has infinite explosion time with probability 1 (cf. Ref. 8).

Evaluating a_{ij}, b_i [by using the assumptions made on $\mu(t)$] is rather lengthy though elementary. We obtain

$$a_{11}=\frac{b}{2}\rho^2, \qquad a_{12}=-a_{21}=\frac{c}{2}\rho, \qquad a_{22}=a+\frac{b}{2} \tag{2.3}$$

$$b_1=\frac{3}{2}b\rho, \qquad b_2=c \tag{2.4}$$

where we set

$$a\equiv\frac{\omega_0^2}{4}S(0), \qquad b\equiv\frac{\omega_0^2}{4}\operatorname{Re}S(2\omega_0), \qquad c\equiv\frac{\omega_0^2}{4}\operatorname{Im}S(2\omega_0) \tag{2.5}$$

$$S(x)\equiv\int_0^{+\infty}R(\xi)\,e^{ix\xi}\,d\xi \tag{2.6}$$

$2S(x)$ being the *power spectral density* of $\mu(t)$. We observe that a_{ij}, b_i are *not* affected by the nonlinearity, in this model.

Computing the c_i's, we get

$$c_1=-\gamma\rho-\frac{w}{4}v_0\rho^3, \qquad c_2=0 \tag{2.7}$$

Therefore

$$L\equiv\frac{b}{2}\rho^2\frac{\partial^2}{\partial\rho^2}+\left(\frac{3}{2}b-\gamma-\frac{w}{4}v_0\rho^2\right)\rho\frac{\partial}{\partial\rho}+\left(a+\frac{b}{2}\right)\frac{\partial^2}{\partial\phi^2}+c\frac{\partial}{\partial\phi} \tag{2.8}$$

With this operator at hand, we are able to describe the time evolution of the expected value of any (bounded) continuous function $Q(\rho, \phi)$ of the limiting process (ρ, ϕ), *and* that of the transition probability density of the solution, in the limit described above. In fact, these are given by the *backward equation*

$$\begin{aligned}\frac{\partial U}{\partial\tau}&=L[U]\\&\equiv\frac{b}{2}\rho^2\frac{\partial^2 U}{\partial\rho^2}+\left(\frac{3}{2}b-\gamma-\frac{w}{4}v_0\rho^2\right)\rho\frac{\partial U}{\partial\rho}\\&\quad+\left(a+\frac{b}{2}\right)\frac{\partial^2 U}{\partial\phi^2}+c\frac{\partial U}{\partial\phi}\\U\,|_{\tau=0}&=Q(\rho, \phi)\end{aligned} \tag{2.9}$$

where $\tau \equiv \varepsilon^2 t$, *and* by the *forward* or *Fokker–Planck equation*

$$\begin{aligned}\frac{\partial P}{\partial \tau} &= L^*[P]\\ &\equiv \frac{b}{2}\frac{\partial^2}{\partial \rho^2}(\rho^2 P) - \frac{\partial}{\partial \rho}\left[\left(\frac{3}{2}b - \gamma - \frac{w}{4}v_0\rho^2\right)\rho P\right]\\ &\quad + \left(a + \frac{b}{2}\right)\frac{\partial^2 P}{\partial \phi^2} - c\frac{\partial P}{\partial \phi}\\ P(\rho, \phi; 0) &= \delta(\rho - \rho(0))\,\delta(\phi - \phi(0))\end{aligned} \tag{2.10}$$

respectively. L^* denotes the *adjoint* of L.

It will be useful sometimes to use the variables (r, ϕ), with $\rho = e^r$, instead of (ρ, ϕ). In these coordinates (2.9), (2.10) take on the form

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \frac{b}{2}\frac{\partial^2 u}{\partial r^2} + \left(b - \gamma - \frac{w}{4}v_0 e^{2r}\right)\frac{\partial u}{\partial r} + \left(a + \frac{b}{2}\right)\frac{\partial^2 u}{\partial \phi^2} + c\frac{\partial u}{\partial \phi}\\ u\,|_{\tau=0} &= q(r, \phi)\end{aligned} \tag{2.9'}$$

where $u(r, \phi; \tau) \equiv U(e^r, \phi; \tau)$, $q(r, \phi) \equiv Q(e^r, \phi)$, and

$$\begin{aligned}\frac{\partial p}{\partial \tau} &= \frac{b}{2}\frac{\partial^2 p}{\partial r^2} + \left(\gamma + \frac{w v_0}{4}e^{2r}\right)\frac{\partial p}{\partial r} + \left(a + \frac{b}{2}\right)\frac{\partial^2 p}{\partial \phi^2}\\ &\quad - c\frac{\partial p}{\partial \phi} + \left(\gamma - \frac{b}{2} + \frac{3}{4}w v_0 e^{2r}\right)p\\ p(r, \phi; 0) &= \delta(r - r(0))\,\delta(\phi - \phi(0))\end{aligned} \tag{2.10'}$$

The advantage of using the coordinates (r, ϕ) is that in the *linear problem* $(w = 0)$ the operator L has *constant coefficients.*

Remark 2.2. We observe that the operator defined in (2.8) can be written

$$L = L_\rho + L_\phi \tag{2.11}$$

where L_ρ does not depend on ϕ and L_ϕ does not depend on ρ. The probabilistic interpretation of this splitting up of the infinitesimal generator is that the two-dimensional Markov process generated by L actually consists of two *independent* one-dimensional Markov processes.

3. STATIONARY DISTRIBUTIONS

A meaningful and important question in the theory of oscillations is to determine whether there exist *stationary* (i.e., *equilibrium*) solutions, that is nontrivial solutions which, eventually, will not vary any more in time. We turn this question to the *statistics* of the solutions. Stationary *distributions* are known *not* to exist in the corresponding linear problem, $w=0$.

Therefore, we set formally $\partial/\partial\tau=0$ in (2.10), and consider the *elliptic* equation thus obtained:

$$\frac{b}{2}\frac{\partial^2}{\partial\rho^2}(\rho^2 P_\infty)-\frac{\partial}{\partial\rho}\left[\left(\frac{3}{2}b-\gamma-\frac{w}{4}v_0\rho^2\right)\rho P_\infty\right]+\left(a+\frac{b}{2}\right)\frac{\partial^2 P_\infty}{\partial\phi^2}-c\frac{\partial P_\infty}{\partial\phi}=0 \tag{3.1}$$

As the coefficients of the operator acting on P_∞ do *not* depend on ϕ, we may look for solutions depending *only* on ρ: The so-obtained equation is still affected by the nonlinearity. This occurrence is due to a possible *competition* "between $\lambda(t)$ and $v(t)\,w(y^\varepsilon)^2$" in the original equation (1.1). Equation (3.1) reduces therefore to

$$\frac{d^2}{d\rho^2}(\rho^2 P_\infty)-\frac{d}{d\rho}\left[\left(3-\frac{2\gamma}{b}-\frac{wv_0}{2b}\rho^2\right)\rho P_\infty\right]=0 \tag{3.2}$$

which has the general solution

$$P_\infty(\rho)=k_1\rho^{\alpha-1}e^{-\eta\rho^2/2}\int_0^\rho \xi^{-(\alpha+1)}e^{\eta\xi^2/2}\,d\xi+k_2\rho^{\alpha-1}e^{-(\eta/2)\rho^2} \tag{3.3}$$

where we set

$$\alpha\equiv 2\left(1-\frac{\gamma}{b}\right),\qquad \eta\equiv\frac{wv_0}{2b} \tag{3.4}$$

and k_1, k_2 are two constants to be determined. Note that $\alpha \gtreqless 0$ for $\gamma \lesseqgtr b$, and $\eta \gtreqless 0$ for $wv_0 \gtreqless 0$. It is lengthy but elementary to show that k_1 must vanish: If we set $P_\infty(\rho)\equiv k_1 y_1(\rho)+k_2 y_2(\rho)$, in fact, it is possible to prove that, otherwise, $k_1 y_1(\cdot)\notin L(R^+)$, whichever the sign of α, η may be (zero included). Moreover, $k_2 y_2(\cdot)\in L(R^+)$, as is required, *only* for $\alpha>0$ (i.e., $\gamma/b<1$) *and* $\eta>0$. We compute k_2 from the *normalization condition*

$$\int_0^\infty P_\infty(\rho)\,d\rho=k_2\int_0^\infty \rho^{1-2\gamma/b}e^{-(wv_0/4b)\rho^2}\,d\rho=1 \tag{3.5}$$

We get

$$k_2 = \frac{2}{\Gamma(1-\gamma/b)} \left(\frac{wv_0}{4b}\right)^{1-\gamma/b} \tag{3.6}$$

and therefore

$$P_\infty(\rho) = \frac{2}{\Gamma(1-\gamma/b)} \left(\frac{wv_0}{4b}\right)^{1-\gamma/b} \rho^{1-2\gamma/b} \exp\left[-\left(\frac{wv_0}{4b}\right)\rho^2\right], \qquad \rho \in R^+ \tag{3.7}$$

(cf. Ref. 10, p. 304).

We now discuss the meaning of the sign of α, η, i.e., the *physical interpretation* of the conditions $\alpha > 0$, $\eta > 0$.

The effect of the *randomness*, which enters the problem through the parameter $b > 0$ is to let the mean-square displacement of the oscillator blow up. A similar effect is produced by the amplification or "negative resistance" due to the system, *whenever* $\gamma < 0$ ($\gamma > 0$ represents a damping). Note that condition $\alpha = 2(1 - \gamma/b) > 0$ is in particular satisfied when $\gamma \leqslant 0$ (recall that for the classical deterministic van der Pol oscillator $\gamma = -1$).

On the other hand, there is a feedback mechanism, due to the *nonlinearity* of the system, that stabilizes the growth of the oscillations, providing a saturation.

Therefore a competition is *possible* among all these phenomena. Without this, *no* stationary solution can exist: There are stationary solutions *if and only if* there is an *actual*, *effective* competition. In particular, the *linear* random oscillator ($\eta = 0$) has *no* stationary solutions as was known. If $b \leqslant \gamma$, on the other hand, there is *no* parametric excitation,[3] and therefore *no* stationary solutions.

4. THE TIME-DEPENDENT PROBLEM

We now turn our attention to the study of the time-dependent solutions of (2.9) [(2.9′)]. As such systems are linear and have coefficients depending only on ρ (on r), we can carry out a Fourier analysis with respect to ϕ, reducing their study to that of the time evolution of single harmonics.

More specifically, we shall discuss system (2.9), for the quantity $U(\rho, \phi; \tau)$. Setting

$$A = 3 - 2\frac{\gamma}{b}, \qquad B = 1 + \frac{2a}{b}, \qquad C = \frac{2c}{b}$$
$$\tau_1 = \frac{b}{2}\tau \tag{4.1}$$

[3] Indeed, $b > \gamma$ is referred to as *the condition for parametric excitation*; cf. Ref. 10, p. 304.

(2.9) becomes

$$\frac{\partial U}{\partial \tau_1} = \rho^2 \frac{\partial^2 U}{\partial \rho^2} + (A - \eta\rho^2)\rho \frac{\partial U}{\partial \rho} + B \frac{\partial^2 U}{\partial \phi^2} + C \frac{\partial U}{\partial \phi} \tag{4.2}$$

$$U(\rho, \phi; 0) = Q(\rho, \phi)$$

with η defined in (3.4). We used the same notation for the function U, after changing τ into τ_1.

Expanding U, Q in Fourier series with respect to ϕ,

$$U(\rho, \phi; \tau_1) = \sum_{m=-\infty}^{+\infty} U_m(\rho, \tau_1)\, e^{im\phi}, \qquad Q(\rho, \phi) = \sum_{m=-\infty}^{+\infty} Q_m(\rho)\, e^{im\phi} \tag{4.3}$$

we get from (4.2)

$$\frac{\partial U_m}{\partial \tau_1} = \rho^2 \frac{\partial^2 U_m}{\partial \rho^2} + (A - \eta\rho^2)\rho \frac{\partial U_m}{\partial \rho} + B_m U_m, \qquad 0 < \rho < \infty, \quad \tau_1 > 0$$

$$U_m(\rho, 0) = Q_m(\rho), \qquad 0 < \rho < \infty \tag{4.4}$$

where

$$B_m = -m^2 B + imC \tag{4.5}$$

We shall consider, typically, $Q(\rho, \phi) = \rho^k e^{ik\phi}$, or $Q(\rho, \phi) = \rho^k$, for a *fixed* k, $k = 1, 2, \ldots$. Therefore

$$Q_m(\rho) = \begin{cases} \rho^k & \text{for } m = k \\ 0 & \text{for } m \neq k \end{cases} \quad (k \text{ fixed} = 1, 2, \ldots) \tag{4.5$'$}$$

or

$$Q_m(\rho) = \begin{cases} \rho^k & \text{for } m = 0 \\ 0 & \text{for } m \neq 0 \end{cases} \quad (k \text{ fixed} = 1, 2, \ldots) \tag{4.5$''$}$$

respectively. In fact, our main goal is to evaluate *the moments*

$$E_{\rho_0}[\rho^k(\tau_1)\, e^{ik\phi(\tau_1)}] = U_k(\rho_0, \tau_1), \qquad k = 1, 2 \tag{4.6$'$}$$

or

$$E_{\rho_0}[\rho^k(\tau_1)] = U_0(\rho_0, \tau_1), \qquad k = 1, 2 \tag{4.6$''$}$$

In (4.6″) we used a different notation for the solution of (4.4) with conditions (4.5″), for convenience.

Note that, by Remark 2.2 and (4.6′), (4.6″),

$$U_k(\rho_0, \tau_1) = E_{\rho_0}[\rho^k(\tau_1)]\, E_{\phi_0}[e^{ik\phi(\tau_1)}] = U_{0,k}(\rho_0, \tau_1)\, E_{\phi_0}[e^{ik\phi(\tau_1)}] \tag{4.7}$$

On the other hand, from (4.2) follows that the process ρ satisfies the Itô equation

$$d\rho = (A - \eta\rho^2)\rho\, d\tau_1 + \sqrt{2}\,\rho\, d\beta(\tau_1) \tag{4.8}$$

$\beta(\cdot)$ being the one-dimensional standard Brownian motion. The process ϕ is a Brownian motion itself, with drift, satisfying the Itô equation

$$d\phi = C\, d\tau_1 + (2B)^{1/2}\, d\beta(\tau_1) \tag{4.9}$$

Therefore [cf. formula (4.23) below]

$$E_{\phi_0}[e^{ik\phi(\tau_1)}] = e^{ik\phi_0} e^{(ikC - Bk^2)\tau_1} \tag{4.10}$$

and from (4.7)

$$U_k(\rho_0, \tau_1) = U_{0,k}(\rho_0, \tau_1)\, e^{ik\phi_0} e^{(ikC - Bk^2)\tau_1} \tag{4.11}$$

In particular, we get

$$|U_k(\rho_0, \tau_1)| = |U_{0,k}(\rho_0, \tau_1)|\, e^{-k^2 B\tau_1} \tag{4.11′}$$

It follows that it will not be necessary to consider the problem (4.4) for U_k, but only that for $U_{0,k}$.

As only the one-dimensional Markov process ρ needs to be studied in detail [cf. Remark 2.2 and (4.8), (4.9)], it is natural to invoke Feller's theory of one-dimensional diffusion.[(2)]

It is convenient to consider the "transformed" process $r = \log \rho$, governed by

$$dr = (\alpha - \eta e^{2r})\, d\tau_1 + \sqrt{2}\, d\beta(\tau_1), \qquad \alpha = A - 1 \tag{4.12}$$

It is easy to check that the boundary $r = -\infty$ ($\rho = 0$) is a *natural boundary*, in the terminology introduced by Feller (see Ref. 2, pp. 487, 516), whichever the sign of α *and* η is. On the other hand, the boundary $r = +\infty$ ($\rho = +\infty$) is an *entrance boundary* when $\eta > 0$ (for any α), and an *exit boundary* when $\eta < 0$ (for any α).

Therefore, when $\eta > 0$ the backward equation has a unique solution, but in the corresponding forward equation uniqueness is lost. The converse happens when $\eta < 0$. Compare also the necessary and sufficient condition for uniqueness given by Hille.[(3)]

Note that, when $\eta < 0$ the (limiting) process *explodes in finite time*, with probability 1. Therefore its moments cannot be defined and, *a fortiori*, convergence of the moments of y^{ε} to those of y becomes meaningless (cf. Ref. 8, p. VI-13, Remark 2).

By setting $U_{0,k} \equiv \rho^k v_{0,k}$, equation (4.4) with $m = k$, together with the IV (4.5″),

$$\frac{\partial U_{0,k}}{\partial \tau_1} = \rho^2 \frac{\partial^2 U_{0,k}}{\partial \rho^2} + (A - \eta\rho^2)\rho \frac{\partial U_{0,k}}{\partial \rho} \qquad (4.13)$$
$$U_{0,k}(\rho, 0) = \rho^k$$

can be transformed into one with a *constant* IV. This can be useful for both numerical and analytical purposes. We obtain

$$\frac{\partial v_{0,k}}{\partial \tau_1} = \rho^2 \frac{\partial^2 v_{0,k}}{\partial \rho^2} + (A + 2k - \eta\rho^2)\rho \frac{\partial v_{0,k}}{\partial \rho} + k(A + k - 1 - \eta\rho^2)\, v_{0,k}$$
$$v_{0,k}(\rho, 0) = 1 \qquad (4.14a)$$

We shall use system (4.14a) in Section 5, for numerical purposes. Here we perform an additional transformation on such a system, rewritten in coordinates (r, τ_1), i.e.,

$$\frac{\partial v_{0,k}}{\partial \tau_1} = \frac{\partial^2 v_{0,k}}{\partial r^2} + (A + 2k - 1 - \eta e^{2r}) \frac{\partial v_{0,k}}{\partial r} + k(A + k - 1 - \eta e^{2r})\, v_{0,k}$$
$$v_{0,k}(r, 0) = 1 \qquad (4.14b)$$

where we wrote $v_{0,k}(r, \tau_1)$ for $v_{0,k}(e^r, \tau_1)$, for short.

We want to state, at this point, existence, uniqueness and some estimates for the solution to problem (4.14b), when $\eta \geq 0$. Note that such a problem involves a partial differential equation over an *unbounded domain* and with *rapidly increasing coefficients*, while (4.14a) is also *degenerate* at $\rho = 0$. This program will be carried out by using the celebrated *Feynman–Kac formula* (cf. Ref. 4, e.g.).

Changing $v_{0,k}$ into $V_{0,k}$, with

$$v_{0,k}(r, \tau_1) = p(r, \eta)\, V_{0,k}(r, \tau_1), \qquad p(r, \eta) \equiv \exp\left(-\frac{\mathcal{A}_k}{2} r + \frac{\eta}{4} e^{2r}\right) \qquad (4.15)$$

where

$$\mathcal{A}_k = A + 2k - 1 \qquad (4.16)$$

in (4.14b), we get

$$\frac{\partial V_{0,k}}{\partial \tau_2} = \frac{1}{2}\frac{\partial^2 V_{0,k}}{\partial r^2} + P(r, \eta)\, V_{0,k}$$
$$V_{0,k}(r, 0) = \frac{1}{p(r, \eta)} = \exp\left(\frac{\mathcal{O}_k}{2} r - \frac{\eta}{4} e^{2r}\right) \tag{4.17}$$

where we set

$$\tau_2 \equiv 2\tau_1$$
$$P(r, \eta) = -\tfrac{1}{8}[\eta^2 e^{4r} - 2(A+1)\,\eta e^{2r} + (A-1)^2] \tag{4.18}$$

Now, $P(r, \eta)$ is *bounded from above*:

$$P(r, \eta) \leqslant \frac{A}{2} \tag{4.19}$$

as is immediately seen. Therefore the solution $V_{0,k}$ to (4.17) can be represented by the Feynman–Kac formula

$$V_{0,k}^{\mathrm{NL}}\left(r, \frac{\tau_2}{2}\right) = E_r\left\{\frac{1}{p(\beta(\tau_2), \eta)} \exp\left[\int_0^{\tau_2} P(\beta(s), \eta)\, ds\right]\right\} \tag{4.20}$$

where $\beta(\cdot)$ denotes the *one-dimensional Brownian motion* on $[0, +\infty)$, starting from $\beta(0) = r$, and $E_r\{\cdot\}$ means taking expected values with respect to the *Wiener measure*. The superscript "NL" stresses the fact that we are considering the *nonlinear* problem ($\eta > 0$). In the corresponding *linear* problem, $\eta = 0$, we have

$$p(r, 0) = \exp(-\mathcal{O}_k r/2), \qquad P(r, 0) = -\tfrac{1}{8}(A-1)^2 \equiv \bar{P} \tag{4.21}$$

and therefore the explicit result

$$V_{0,k}^{\mathrm{L}}\left(r, \frac{\tau_2}{2}\right) = E_r\left\{\frac{1}{p(\beta(\tau_2), 0)} \exp\left[\int_0^{\tau_2} P(\beta(s), 0)\, ds\right]\right\}$$
$$= E_r\left\{\exp\left[\frac{\mathcal{O}_k}{2}\beta(\tau_2)\right]\right\} e^{\bar{P}\tau_2}$$
$$= \exp\left(\frac{\mathcal{O}_k}{2} r\right) \exp\left[\left(\frac{\mathcal{O}_k^2}{8} + \bar{P}\right)\tau_2\right] \tag{4.22}$$

Here "L" stands for linear and we used the formula

$$E_x\{e^{h\beta(t)}\} = e^{hx} e^{h^2 t/2} \tag{4.23}$$

valid for any constant h.

From (4.20), (4.18), and (4.21) we get

$$V_{0,k}^{\mathrm{NL}}\left(r, \frac{\tau_2}{2}\right) = e^{\bar{P}\tau_2} E_r \left\{ \exp\left[\frac{\mathcal{A}_k}{2} \beta(\tau_2) - \frac{\eta}{4} e^{2\beta(\tau_2)} \right] \right.$$
$$\left. \times \exp\left\{ -\frac{1}{8}\eta \int_0^{\tau_2} e^{2\beta(s)} [\eta e^{2\beta(s)} - 2(A+1)]\, ds \right\} \right\} \tag{4.24}$$

For $\eta \geqslant 0$ we get the estimate

$$0 \leqslant V_{0,k}^{\mathrm{NL}}\left(r, \frac{\tau_2}{2}\right) \leqslant e^{\bar{P}\tau_2} E_r \left\{ \exp\left[\frac{\mathcal{A}_k}{2} \beta(\tau_2) \right] \right\} \exp\left[\frac{1}{8}(A+1)^2 \tau_2 \right]$$
$$= \exp\left(\frac{\mathcal{A}_k}{2} r \right) \exp\left[(\mathcal{A}_k^2 + 4A) \frac{\tau_2}{8} \right] \tag{4.25}$$

as $-\frac{1}{8}\eta e^{2x}[\eta e^{2x} - 2(A+1)] \leqslant \frac{1}{8}(A+1)^2$, and using formula (4.23). Therefore

$$0 \leqslant V_{0,k}^{\mathrm{NL}}(r, \tau_1) \leqslant \exp\left(\frac{\mathcal{A}_k}{2} r \right) \exp\left[(\mathcal{A}_k^2 + 4A) \frac{\tau_1}{4} \right] \tag{4.26}$$

Recalling (4.22), we may also write

$$0 \leqslant V_{0,k}^{\mathrm{NL}}(r, \tau_1) \leqslant V_{0,k}^{\mathrm{L}}(r, \tau_1) \exp\left[\frac{(A+1)^2}{4} \tau_1 \right] \tag{4.27}$$

Going back to $v_{0,k}^{\mathrm{NL}}$, $v_{0,k}^{\mathrm{L}}$, we obtain from (4.15) and (4.26)

$$0 \leqslant v_{0,k}^{\mathrm{NL}}(r, \tau_1) \leqslant \exp\left(\frac{\eta}{4} e^{2r} \right) \exp\left[(\mathcal{A}_k^2 + 4A) \frac{\tau_1}{4} \right] \tag{4.28}$$

Similarly, we get

$$v_{0,k}^{\mathrm{L}}(r, \tau_1) = \exp\left[\left(2\bar{P} + \frac{\mathcal{A}_k^2}{4} \right) \tau_1 \right]$$
$$= \exp[k(A+k-1)\,\tau_1] \tag{4.29}$$

Finally, it can be useful to *compare* $v_{0,k}^{\mathrm{NL}}$ with $v_{0,k}^{\mathrm{L}}$:[4]

$$0 \leqslant v_{0,k}^{\mathrm{NL}}(r, \tau_1) \leqslant v_{0,k}^{\mathrm{L}}(r, \tau_1) \exp\left(\frac{\eta}{4} e^{2r} \right) \exp\left\{ \frac{(A+1)^2}{4} \tau_1 \right\} \tag{4.30}$$

[4] B. White pointed me out that, starting from (4.8), the estimate (4.30) could be improved by getting $v_{0,k}^{\mathrm{NL}} \leqslant v_{0,k}^{\mathrm{L}}$.

Observe that (4.29) can be derived directly from (4.14b) with $\eta = 0$: as the IV as well as the coefficients are independent of r, looking for solutions also independent of r, we just have to solve

$$\frac{\partial v_{0,k}}{\partial \tau_1} = k(A + k - 1)\, v_{0,k}, \qquad v_{0,k}(r, 0) = 1$$

which gives (4.29), as

$$2\bar{P} + \frac{\alpha_k^2}{4} = k(A + k - 1)$$

recalling (4.21) and (4.16).

We make some comments, finally, about the (time) growth rates of $U_k(\rho_0, \tau_1)$, as by (4.11′)

$$|U_k(\rho_0, \tau_1)| = \rho_0^k v_{0,k}(\rho_0, \tau_1)\, e^{-k^2 B \tau_1}$$

In the *linear* case, recalling (4.29) and (4.1), we get

$$\begin{aligned} k(A + k - 1) - k^2 B &= -k[k(B-1) - (A-1)] \\ &= -2k\left[\left(\frac{ka}{b} - 1\right) + \frac{\gamma}{b}\right] \end{aligned} \tag{4.31}$$

For some k, $k = 1, 2, \ldots$, we may have growth, but certainly for k large enough there will be decay. When, e.g., $a + \gamma \geqslant b$, we have decay for all $k = 1, 2, 3, \ldots$. This condition is satisfied, for instance, when $\gamma \geqslant 0$, as $a \geqslant b$, but also for $\gamma < 0$, $|\gamma| < a - b$. In any case we get the bound $(1/4)[(A-1)^2/(B-1)] = (b/2a)(1 - \gamma/b)^2$ for all the growth rates:

$$0 \leqslant v_{0,k}^L(r, \tau_1) \leqslant \exp\left[\frac{b}{2a}\left(1 - \frac{\gamma}{b}\right)^2 \tau_1\right], \qquad k = 1, 2, 3, \ldots \tag{4.32}$$

Similarly, in the *nonlinear* case we have, from (4.30), the time growth rate

$$\tfrac{1}{4}(A + 1)^2 + \text{linear growth rate}$$

the linear growth rate being given by (4.31). Therefore it is estimated by

$$\frac{1}{4}\left[(A + 1)^2 + \frac{(A-1)^2}{B - 1}\right]$$

uniformly, in k, $k=1, 2, 3,\ldots$. Therefore we obtain

$$0 \leqslant v_{0,k}^{\mathrm{NL}}(r, \tau_1) \leqslant \exp\left(\frac{\eta}{4} e^{2r}\right) \times \exp\left\{\left[1+2\left(1-\frac{\gamma}{b}\right)+\left(1+\frac{b}{2a}\right)\left(1-\frac{\gamma}{b}\right)^2\right]\tau_1\right\} \tag{4.33}$$

uniformly in k, $k=1, 2, 3,\ldots$.

It is also possible to obtain estimates for the derivatives $\partial v_{0,k}^{\mathrm{NL}}/\partial r$, $\partial^2 v_{0,k}^{\mathrm{NL}}/\partial r^2$, or $v_{k,r}^{\mathrm{NL}}$, $v_{k,rr}^{\mathrm{NL}}$, where $v_k^{\mathrm{NL}} \equiv U_k/\rho^k$, the solution to (4.4), (4.5′), $m=k$ [cf. (4.11)]. We describe the procedure and omit the details. By differentiating in (4.14a), we obtain a system for $\partial v_{0,k}^{\mathrm{NL}}(r, \tau_1)/\partial r$. The equation here will be inhomogeneous, but the inhomogeneity term will depend only on $v_{0,k}^{\mathrm{NL}}$, and this, in view of (4.28), can be considered as "known" for the purpose of estimating $\partial v_{0,k}^{\mathrm{NL}}/\partial r$. The IV is zero. Then we represent the relevant Green's function for the corresponding homogeneous problem by means of the Feynman–Kac formula, and obtain therefore the solution to the full inhomogeneous problem as well. It results in the estimate for $v_{k,r}^{\mathrm{NL}}$,

$$|v_{k,r}^{\mathrm{NL}}(r, \tau_1)| \leqslant c_1 \exp\left(\frac{\eta}{4} e^{2r} + 2r\right) e^{c_2\tau_1}(1-e^{-c_3\tau_1}) \tag{4.34}$$

where

$$c_1 \equiv \frac{k\eta}{2(k+1)}, \qquad c_2 \equiv \left(\frac{\mathfrak{A}_k}{2}+2\right)^2 - (A+k^2B-2), \qquad c_3 \equiv 4(k+1) \tag{4.35}$$

$\mathfrak{A}_k$ being defined in (4.16).

By proceeding in a similar way, we can derive an estimate for $v_{k,rr}^{\mathrm{NL}}$ as well:

$$|v_{k,rr}^{\mathrm{NL}}(r, \tau_1)| \leqslant 4\eta \exp\left(\frac{\eta}{4} e^{2r} + 2r\right) \exp\{-[3(A-1)+k^2B-9]\,\tau_1\} \times \left[\frac{k\eta}{2} e^{2r}\left(\frac{e^{d_1\tau_1}-1}{d_1} - \frac{e^{d_2\tau_2}-1}{d_2}\right) + k\frac{(e^{d_3\tau_1}-1)}{d_3}\right] \tag{4.36}$$

where we set, for short,

$$\begin{aligned} d_1 &\equiv \frac{(A-1)^2}{2} + 2(k+4)(A-1) + 2k^2 + 12k + 8 \\ d_2 &\equiv d_1 - c_3 = \frac{1}{2}(A-1)^2 + 2(k+4)(A-1) + 2k^2 + 8k + 4 \\ d_3 &\equiv \frac{1}{2}(A-1)^2 + 2(k+3)(A-1) + 2k^2 + 4k - 4 \end{aligned} \tag{4.37}$$

The estimates (4.34), (4.36) show, in particular, that $v_{k,r}^{NL}(r, \tau_1)$, $v_{k,rr}^{NL}(r, \tau_1)$ are infinitesimal as $r \to -\infty$, for $\tau_1 \in [0, T]$, $\forall T > 0$.

5. NUMERICAL TREATMENT

In this Section we describe the results of the numerical treatment of equation (4.14a) that we performed to obtain the time behavior of the first two moments of the displacement of the oscillator corresponding to the model equation (1.1).

Equation (4.14a) actually describes the time evolution of the moments of the *limiting process*, and we shall consider them as *an approximation* of the former quantities.

In order to carry out the computations in practice, we must choose a *correlation function*, for example, the exponential

$$R(\xi) = R_0 \exp(-|\xi|/\xi_0) \tag{5.1}$$

where $\xi_0 > 0$ represents some *correlation time*. Then the *power spectral density* can be obtained from (2.6):

$$S(x) = \frac{R_0 \xi_0 (1 + i x \xi_0)}{1 + x^2 \xi_0^2}$$

and therefore, from (2.5) we get

$$a = \frac{\omega_0^2}{4} R_0 \xi_0, \qquad b = \frac{\omega_0^2}{4} \frac{R_0 \xi_0}{1 + 4\omega_0^2 \xi_0^2}, \qquad c = \frac{\omega_0^2}{4} \frac{2(R_0 \xi_0)(\omega_0 \xi_0)}{1 + 4\omega_0^2 \xi_0^2}$$

We need only the *nondimensional* quantities

$$\begin{aligned} \frac{a}{b} &= 1 + (2\omega_0 \xi_0)^2, & \frac{c}{b} &= 2\omega_0 \xi_0 \\ \frac{\gamma}{b} &= \frac{8}{R_0} \frac{\gamma}{\omega_0} \frac{1 + (2\omega_0 \xi_0)^2}{2\omega_0 \xi_0}, & \eta &= \frac{w v_0}{2b} \end{aligned} \tag{5.2}$$

However, the only *independent* parameters to be assigned are $\omega_0 \xi_0$, γ/ω_0, R_0, and η. Also the IV $\rho(0) = \rho_0$ and ε must be considered as parameters: As for the latter, observe that there are *two time scales*, according to the times t and $\tau_1 \equiv (b/2)\, \varepsilon^2 t$.

In handling numerically systems such as (4.14a), we notice that they should be integrated over an *unbounded domain* $\mathscr{D}_{\infty,T} \equiv \{(\rho, \tau_1)\colon 0 < \rho < \infty, 0 < \tau_1 < T\}$. In practice, of course, we have "to cut" such a domain, to get a bounded one, say $\mathscr{D}_{\rho_{\max},T} \equiv \{(\rho, \tau_1): \ 0<\rho<\rho_{\max}, \ 0<\tau_1<T\}$. An additional boundary condition is thus required at $\rho=\rho_{\max}$. In order to get such a condition, we exploit the *symmetry* of both the equation and the IV in (4.14a). This implies that

$$\left.\frac{\partial v_{0,k}}{\partial \rho}\right|_{\rho=0}=0 \tag{5.3}$$

provided that $\lim_{\rho \to 0+} \partial v_{0,k}/\partial \rho$ exists. Indeed, cf. (4.34), recalling that $\rho = e^r$. Moreover, as changing ρ into σ, $\sigma = 1/\rho$, takes system (4.14a) into

$$\begin{aligned}\frac{\partial v_{0,k}}{\partial \tau_1} &= \sigma^2 \frac{\partial^2 v_{0,k}}{\partial \sigma^2} + \left[2 - A - 2k + \frac{\eta}{\sigma^2}\right] \sigma \frac{\partial v_{0,k}}{\partial \sigma} \\ &\quad + \left[k(A+k-1) - \frac{k\eta}{\sigma^2}\right] v_{0,k}, \qquad 0<\sigma<\infty, \quad \tau_1>0 \\ v_{0,k}&\left(\frac{1}{\sigma}, 0\right) = 1, \qquad 0<\sigma<\infty\end{aligned} \tag{5.4}$$

if we assume $v_{0,k}$ to be differentiable at $\sigma=0$ (i.e., at $\rho=\infty$), we obtain, again by *symmetry*, the boundary value $\partial v_{0,k}/\partial \sigma \mid_{\sigma=0} = 0$. We impose *such* a condition on the "*artificial*" boundary $\rho = \rho_{\max}$:

$$\left.\frac{\partial v_{0,k}}{\partial \rho}\right|_{\rho=\rho_{\max}}=0 \tag{5.5}$$

Then we solve numerically system (4.14a), completed by conditions (5.3), (5.5). The final goal is to compute

$$\begin{aligned}E_{\rho_0}\{y(t)\} &= \rho_0 \operatorname{Re}\{v_1(\rho_0, \tau_1)\} \cos \omega_0 t \\ &\quad - \rho_0 \operatorname{Im}\{v_1(\rho_0, \tau_1)\} \sin \omega_0 t \qquad \left(\tau_1 \equiv \frac{b}{2} \varepsilon^2 t\right)\end{aligned} \tag{5.6}$$

from (4.14a) with $k=1$, and recalling (4.11) and the transformation $U_{0,k} \equiv \rho^k v_{0,k}$, $U_k \equiv \rho^k v_k$. Similarly, for $k=2$

$$\begin{aligned}E_{\rho_0}\{y^2(t)\} &= \tfrac{1}{2}[\rho_0^2 \operatorname{Re}\{v_2(\rho_0, \tau_1)\} \cos(2\omega_0 t) \\ &\quad - \rho_0^2 \operatorname{Im}\{v_2(\rho_0, \tau_1)\} \sin(2\omega_0 t)] + \tfrac{1}{2}\rho_0^2 v_{0,2}(\rho_0, \tau_1)\end{aligned} \tag{5.7}$$

The numerical implementation of systems (4.14a) was done by using an *implicit* scheme of *finite differences*, namely the Crank–Nicholson scheme.

The choice of ρ_{max} must be such that we can compute good, accurate values of $v_{0,k}^{NL}$ *at* $\rho = \rho_0$, up to some time $\tau_{max} = T$, within a certain fixed error. In fact, as $\rho = \rho_{max}$ approximates $\rho = \infty$, by reducing it we may expect loss of accuracy, as the time integration proceeds.

For a given $\rho_0 \equiv \rho(0)$, we chose N_0 and $\Delta\rho = \rho_0/(N_0 - 1)$, with $\Delta\rho$ sufficiently small, for the purpose of accuracy. Then we determined N_ρ and $\rho_{max} = (N_\rho - 1)\,\Delta\rho$, together with T, by experimentation. As we were content to get good results up to $\rho = \rho_0$, we reduced N_ρ in such a way that the results were appreciably unchanged (within an error of 10^{-5}) up to $\rho = \rho_0$. Up to $N_\tau = 1200$ (corresponding to $T = N_\tau\,\Delta\tau$, $\Delta\tau$ as below), this was largely true when $\rho_0 = 1$, $N_0 = 20$, for $N_\rho = 60$. Therefore we used a space mesh $\Delta\rho \simeq 0.0526$. However, as this experiment was conducted for a single set of parameters, we chose N_ρ appreciably larger, $N_\rho = 80$, so that $\rho_{max} \simeq 4.1578$. Also, the quantities we computed were essentially zero before arriving at $N_\tau = 1200$ time steps.

By reducing the time mesh $\Delta\tau$ from 0.4×10^{-3} to 0.4×10^{-4}, no appreciable improvement was observed in computing the first moments. The same happened reducing further $\Delta\tau$ to 0.1×10^{-4}, for the second moments. We used $\Delta\tau = 0.4 \times 10^{-4}$ throughout the computations. It was noticed that $\Delta\tau$ should be reduced when the nonlinearity parameter η is increased more and more *and* the integration is extended over long time ranges, to avoid numerical instability.

Other parameters used were $R_0 = 1$, $\omega_0\xi_0 = 1$, and $\varepsilon = 0.1$, while γ/b and $\eta \equiv v_0 w/2b$ were given several values.

We plotted the results in Figs. 1–9. In Figs. 1A, 2A, and 3A we plotted the modulus of the first moment, that of the time-fluctuating part in the second moment, and the *phase-independent* part of the second moment, respectively, for $\gamma/b = 0.1$, $\eta \equiv v_0 w/2b = 2$. These quantities are v_1^{NL}, v_2^{NL}, $v_{0,2}^{NL}$, in the notation of Sections 4 and 5 above. The quantities for the corresponding *linear* problem are also plotted, for comparison.

In Fig. 4 we show the second moment (obtained from the quantities plotted in Fig. 2A, 3A), as well as the corresponding quantity for the linear problem. The *modulation* of the unperturbed oscillations by the randomness is also evident.

The function represented in Fig. 4(a) tends to a nontrivial limit, as $(b/2)\,\varepsilon^2 t \to \infty$, since $\gamma/b < 1$ *and* $\eta > 0$ correspond to a case in which *stationary solutions* occur (cf. Section 3). The function represented in Fig. 4(b) blows up in the limit, as was known for the linear random oscillator (cf. Ref. 6). We note also that the first moments are less damped than the second ones (Figs. 1A, 2A).

In Fig. 1B, 2B, and 3B we plotted quantities like those in Figs. 1A, 2A, 3A, respectively, but for $\gamma/b = 0.8$, $\eta = 2$. The *phase* is also shown, while in

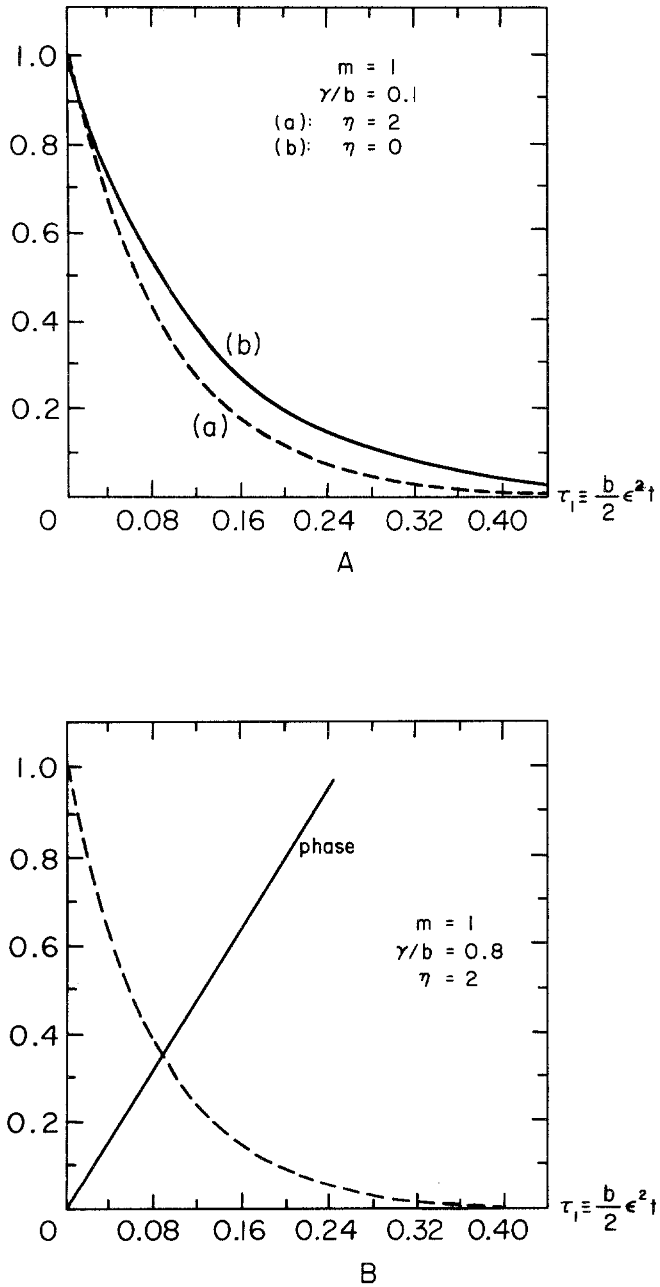
1.0
0.8
0.6
0.4
0.2
0
0.08
0.16
0.24
0.32
0.40
m = 1
γ/b = 0.1
(a): η = 2
(b): η = 0
(a)
(b)
$\tau_1 \equiv \frac{b}{2}\epsilon^2 t$
A
1.0
0.8
0.6
0.4
0.2
0
0.08
0.16
0.24
0.32
0.40
phase
m = 1
γ/b = 0.8
η = 2
$\tau_1 \equiv \frac{b}{2}\epsilon^2 t$
B

Fig. 1.

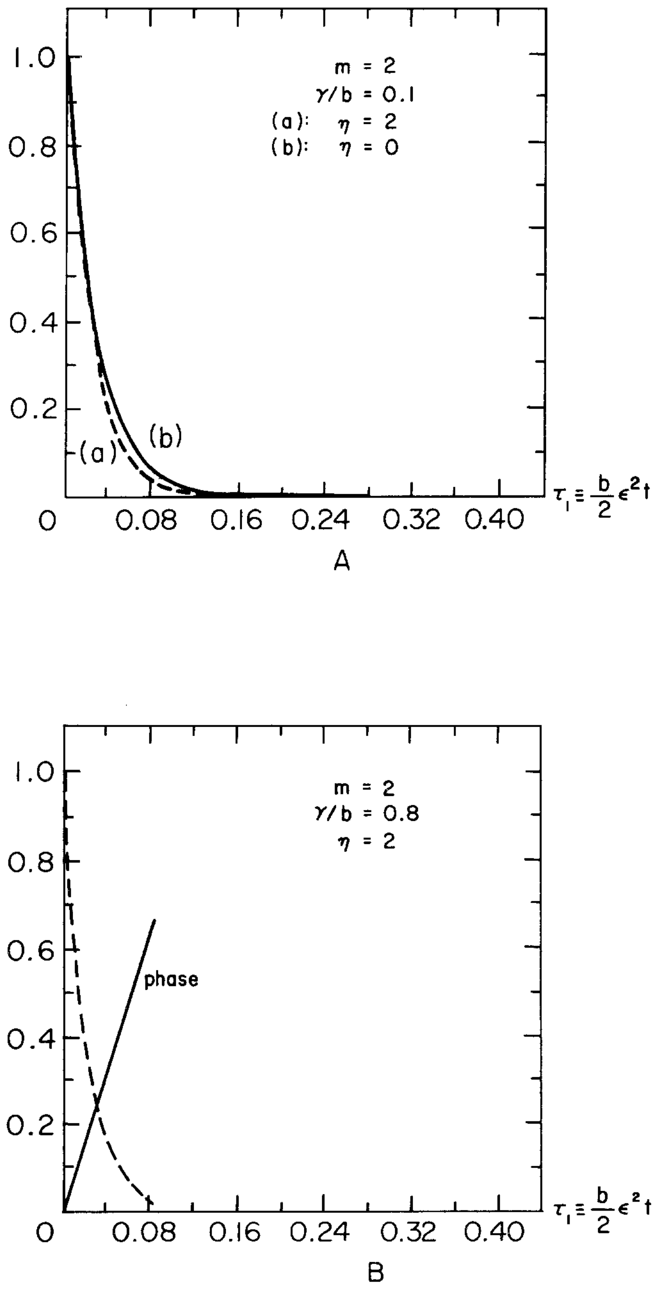
1.0
0.8
0.6
0.4
0.2
0
0.08
0.16
0.24
0.32
0.40
m = 2
γ/b = 0.1
(a): η = 2
(b): η = 0
(a)
(b)
τ₁ ≡ b/2 ε² t
A
1.0
0.8
0.6
0.4
0.2
0
0.08
0.16
0.24
0.32
0.40
m = 2
γ/b = 0.8
η = 2
phase
τ₁ ≡ b/2 ε² t
B

Fig. 2.

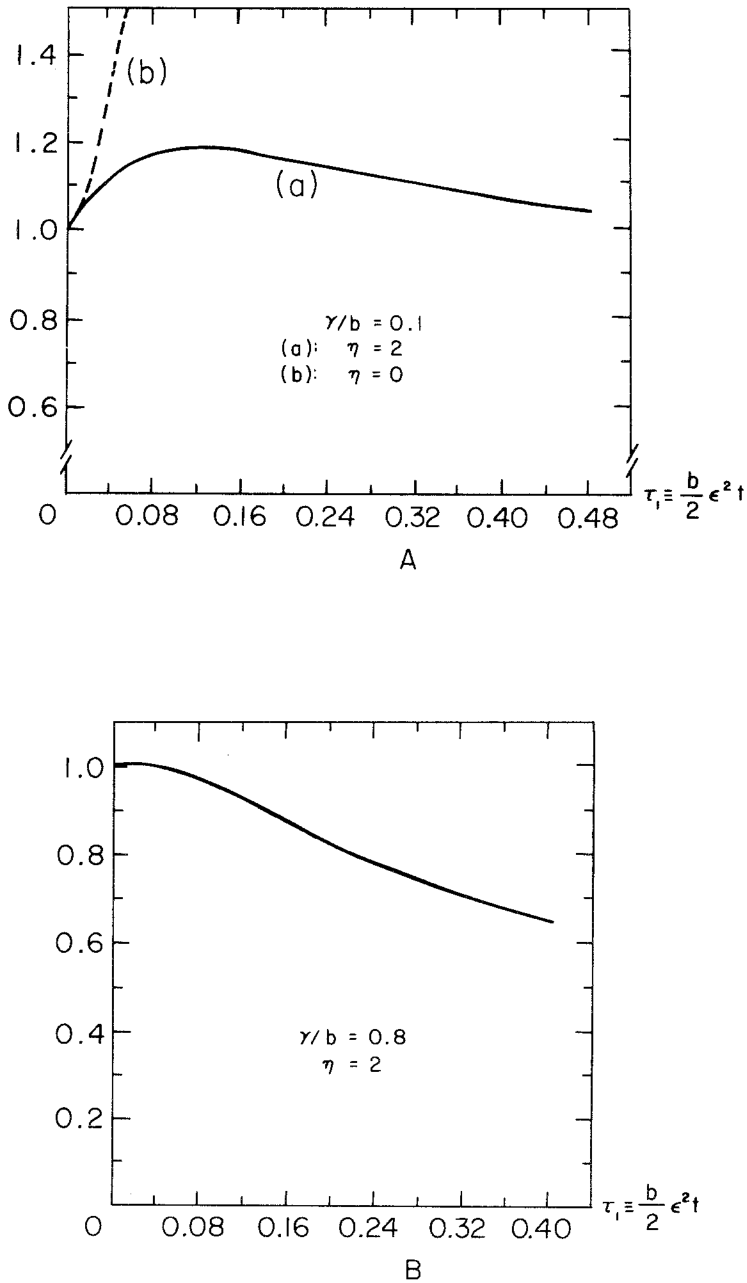
1.4
1.2
1.0
0.8
0.6
(b)
(a)
γ/b = 0.1
(a): η = 2
(b): η = 0
0
0.08
0.16
0.24
0.32
0.40
0.48
τ₁ ≡ b/2 ε²t
A
1.0
0.8
0.6
0.4
0.2
γ/b = 0.8
η = 2
0
0.08
0.16
0.24
0.32
0.40
τ₁ ≡ b/2 ε²t
B

Fig. 3.

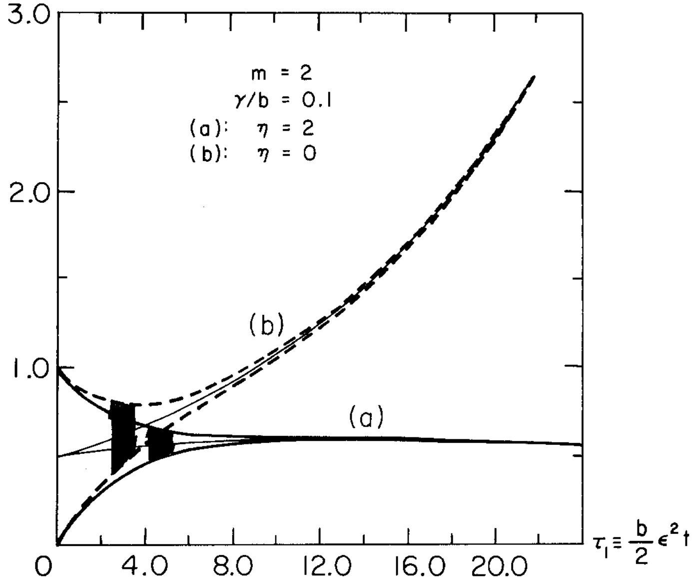
m = 2
γ/b = 0.1
(a): η = 2
(b): η = 0
(b)
(a)
3.0
2.0
1.0
0
4.0
8.0
12.0
16.0
20.0
$\tau_1 \equiv \frac{b}{2}\epsilon^2 t$

Fig. 4.

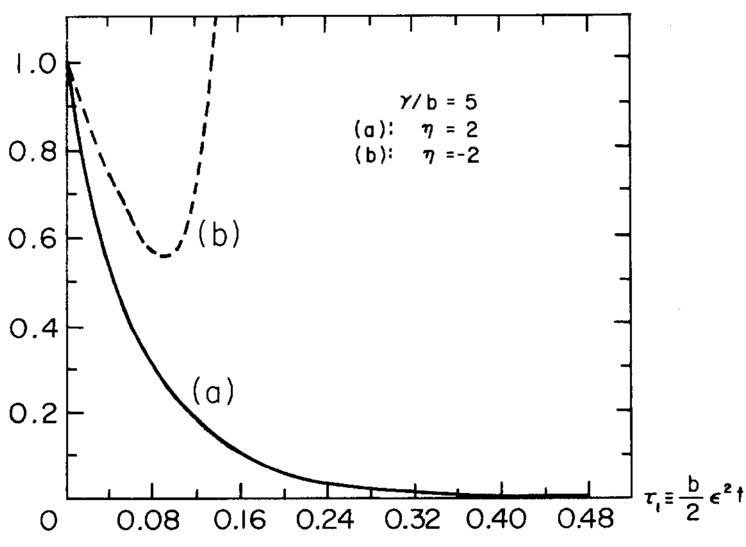
γ/b = 5
(a): η = 2
(b): η =-2
(b)
(a)
1.0
0.8
0.6
0.4
0.2
0
0.08
0.16
0.24
0.32
0.40
0.48
$\tau_1 \equiv \frac{b}{2}\epsilon^2 t$

Fig. 5.

general we did not draw it, being of minor importance. There are, in fact, *two time scales* in the oscillator we have been studying, one according to the deterministic oscillations, and one characterizing the random fluctuations. They can be compared by the ratio

$$\kappa := \frac{\omega_0 t}{\tau_1} = \frac{2\omega_0}{b\varepsilon^2} \tag{5.8}$$

(τ_1 is nondimensional). Recalling (5.2) and using $R_0 = 1$, $\omega_0 \xi_0 = 1$, to compute $\omega_0/b = (\gamma/b)/(\gamma/\omega_0)$, we get $\kappa = 40/\varepsilon^2$. Choosing $\varepsilon = 0.1$ we get

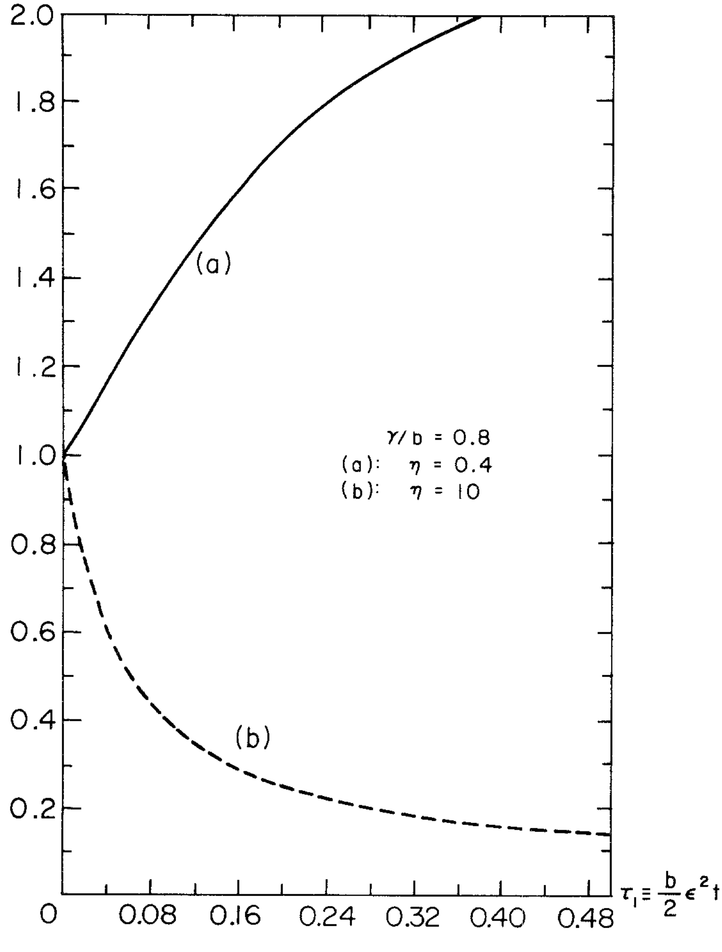

Fig. 6.

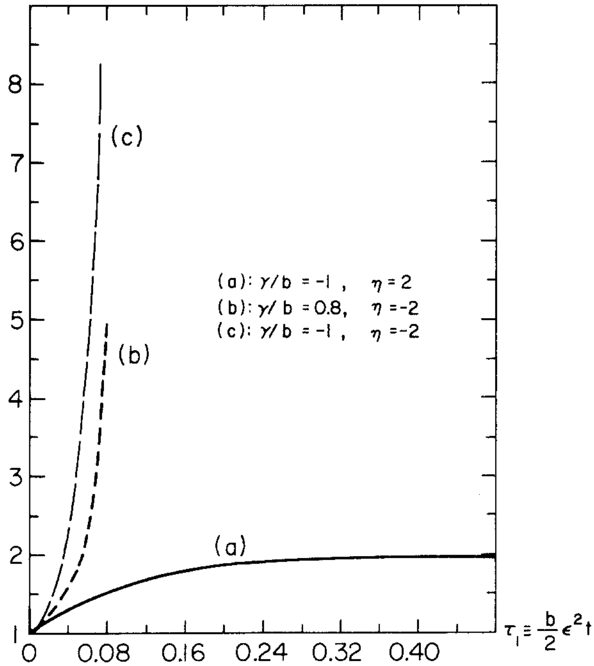
8
7
6
5
4
3
2
1
0
0.08
0.16
0.24
0.32
0.40
(c)
(b)
(a)
(a): γ/b = -1 , η = 2
(b): γ/b = 0.8, η = -2
(c): γ/b = -1 , η = -2
τ₁ ≡ b/2 ε² t

Fig. 7.

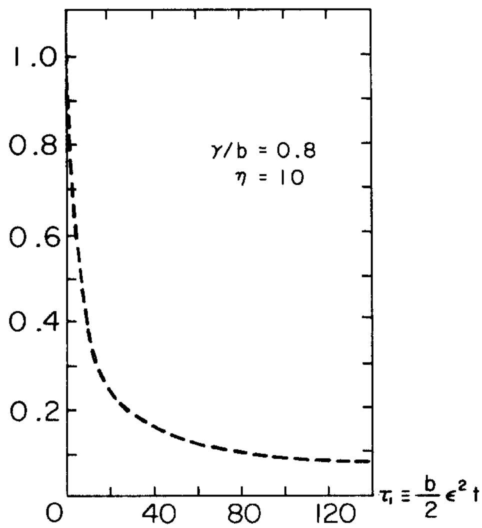
1.0
0.8
0.6
0.4
0.2
0
40
80
120
γ/b = 0.8
η = 10
τ₁ ≡ b/2 ε² t

Fig. 8.

$\kappa = 4000$, which means that 4000 deterministic oscillations take place on the time scale characteristic of the random fluctuations and therefore the phase changes due to the randomness are negligible.

In Figs. 5–9, we show the *phase-independent* part of the second moment, $v_{0,2}$, for several values of the parameters γ/b and η. Figure 5(a) corresponds to $\gamma/b = 5$, $\eta = 2$. Recalling that there exist (nontrivial) *stationary distributions if and only if* $\gamma/b < 1$ *and* $\eta > 0$ (cf. Section 3), there is numerical evidence that, indeed, in such case they do *not* occur.

The graph (b), in Fig. 5 represents the solution which obtains by setting $\eta = -2$ in the previous case. It must be clear that, when $\rho_{\max} \to \infty$, this will *not* give the solution to the original problem with $\eta = -2$: In fact, for $\eta < 0$ uniqueness is lost, as was observed above.

In Fig. 6, we present two cases with $\gamma/b = 0.8$: (a) $\eta = 0.4$ and (b) $\eta = 10$, where stationary distributions do occur. However this does not become clear in a relatively short range.

In Fig. 7 things are much more transparent: In case (a), $\gamma/b = -1$ (cf. the classical van der Pol oscillator), $\eta = 2$, there is a *clear* numerical evidence of the occurrence of stationary solutions. Fig. 7, (b) corresponds to $\gamma/b = 0.8$, $\eta = -2$, and Fig. 7, (c) to $\gamma/b = -1$, $\eta = -2$, but, as was observed before, the cases with $\eta < 0$ do *not* correspond to cases of interest.

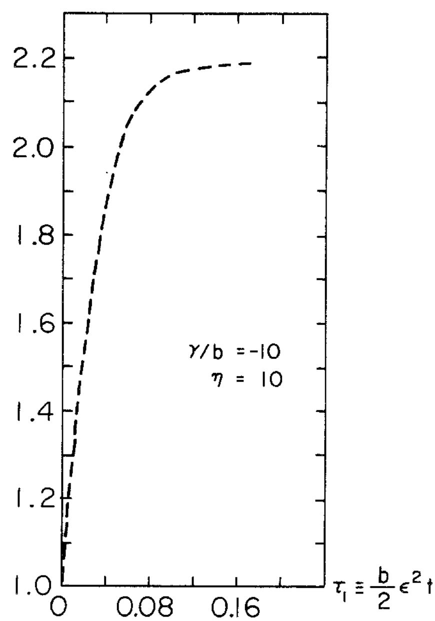

Fig. 9.

Figure 8 shows the behavior of $v_{0,2}$ for $\gamma/b = 0.8$, $\eta = 10$, and finally Fig. 9 does the same for $\gamma/b = -10$, $\eta = 10$.

From the graphs in Figs. 6–9 we can see that stronger nonlinearity η, with $\eta > 0$, "stabilizes" more (i.e., faster) the behavior. In fact it is the nonlinearity which "saturates" the oscillations. On the other hand, negative values of γ satisfy the condition required to have stationary distributions (for what concerns γ), and have the effect of raising the curves, (compare Fig. 7(b) with Fig. 7(c), *and* Fig. 8 with Fig. 9). In fact, $\gamma < 0$ represents an amplification.

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REFERENCES

1. G. Blankenship and G. C. Papanicolaou, Stability and control of stochastic systems with wide-band noise disturbances. I, *SIAM J. Appl. Math.* **34**:437–476 (1978).
2. W. Feller, The parabolic differential equations and the associated semi-groups of transformations, *Ann. Math.* **55**:468–519 (1952).
3. E. Hille, Le probabilités continues en chaîne, *C. R. Acad. Sci. Paris* **230**:34–35 (1950).
4. M. Kac, On distributions of certain Wiener functionals, *Trans. Amer. Math. Soc.* **65**:1–13 (1949).
5. R. Z. Khas'minskii, A limit theorem for the solutions of differential equations with a random right hand side, *Theor. Probability Appl.* **11**:390–406 (1966).
6. G. Papanicolaou and J. B. Keller, Stochastic differential equations with applications to random harmonic oscillators and wave propagation in random media, *SIAM J. Appl. Math.* **21**:287–305 (1971).
7. G. Papanicolaou, Wave propagation in a one dimensional random medium, *SIAM J. Appl. Math.* **21**:13–18 (1971).
8. G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan, Martingale approach to some limit theorems, in *Statistical Mechanics, Dynamical Systems and the Duke Turbulence Conference*, D. Ruelle ed., Duke Univ. Math. Series, Vol. 3 (Duke University Press, Durham, North Carolina, 1977).
9. J. J. Stoker, *Nonlinear Vibrations in Mechanical and Electrical Systems* (Wiley-Interscience, New York, 1950).
10. R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. II (Gordon and Breach, New York, 1967).